Solutions for Calculus BC More Taylor Practice

1. (a) $P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$
$= 1 + 4(x-3) + 3(x-3)^2 + 2(x-3)^3$
$f(3.2) \approx P_3(3.2) = 1.936$

(b) Since the Taylor series for f' can be obtained by term-by-term differentiation of the Taylor Series for f, the second order Taylor polynomial for f' at $x = 3$ is $4 + 6(x-3) + 6(x-3)^2$. Evaluated at $x = 2.7$, $f'(2.7) \approx 2.74$.

(c) It underestimates the values, since $f''(3) = 6$, which means the graph of f is concave up near $x = 3$.

2. (a) Since the constant term is $f(4)$, $f(4) = 7$. Since $-2 = \frac{f'''(4)}{3!}$, $f'''(4) = -12$.

(b) Note that $P_4'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$. The second degree polynomial for f' at $x = 4$ is given by the first three terms of this expression, namely $-3 + 10(x-4) - 6(x-4)^2$. Evaluating at $x = 4.3$, $f'(4.3) \approx -0.54$.

(c) The fourth order Taylor polynomial for $g(x)$ at $x = 4$ is
$\int_4^x [7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3]\, dx$
$= \left[7t - \frac{3}{2}(t-4)^2 + \frac{5}{3}(t-4)^3 - \frac{1}{2}(t-4)^4\right]_4^x$
$= 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$

(d) No. One would need the entire Taylor series for $f(x)$, and it would have to converge to $f(x)$ at $x = 3$.

3. (a) Use the Maclaurin series for $\sin x$ given at the end of Section 9.2.

$5\sin\left(\frac{x}{2}\right)$
$= 5\left[\frac{x}{2} - \frac{(x/2)^3}{3!} + \frac{(x/2)^5}{5!} - \cdots + (-1)^n \frac{(x/2)^{2n+1}}{(2n+1)!} + \cdots\right]$
$= \frac{5x}{2} - \frac{5x^3}{48} + \frac{x^5}{768} - \cdots + (-1)^n \frac{5}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1} + \cdots$

(b) The series converges for all real numbers, according to the Ratio Test:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5}{(2n+3)!}\left|\frac{x}{2}\right|^{2n+3} \cdot \frac{(2n+1)!}{5}\left|\frac{2}{x}\right|^{2n+1}$
$= \lim_{n\to\infty}\frac{|x/2|^2}{(2n+3)(2n+2)} = 0$

4. (a) Substitute $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ given at the end of Section 9.2.
$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \cdots + (2x)^n + \cdots$
$= 1 + 2x + 4x^2 + 8x^3 + \cdots + (2x)^n + \cdots$

(b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The series for $\frac{1}{1-t}$ is known to converge for $-1 < t < 1$, so by substituting $t = 2x$, we find the resulting series converges for $-1 < 2x < 1$.

(c) $f\left(-\frac{1}{4}\right) = \frac{2}{3}$, so one percent is approximately 0.0067. It takes 7 terms (up through degree 6). This can be found by trial and error. Also, for $x = -\frac{1}{4}$, the series is the alternating series $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$. If you use the Alternating Series Estimation Theorem, it shows that 8 terms (up through degree 7) are sufficient since $\left|-\frac{1}{2}\right|^8 < 0.0067$. It is also a geometric series, and you could use the remainder formula for a geometric series to determine the number of terms needed. (See Example 2 in Section 9.3.)

5. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n n^n}$
$= \lim_{n\to\infty}\frac{|x|(n+1)^{n+1}}{(n+1)n^n}$
$= |x| \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = |x|e$

The series converges for $|x|e < 1$, or $|x| < \frac{1}{e}$, so the radius of convergence is $\frac{1}{e}$.

(b) $f\left(-\frac{1}{3}\right) \approx -\frac{1}{3} \cdot \frac{1}{1} + \left(-\frac{1}{3}\right)^2 \cdot \frac{2^2}{2!} + \left(-\frac{1}{3}\right)^3 \cdot \frac{3^3}{3!}$
$= -\frac{1}{3} + \frac{2}{9} - \frac{1}{6}$
$= -\frac{5}{18} = -0.278$

(c) By the Alternating Series Estimation Theorem the error is no more than the magnitude of the next term, which is $\left|\left(-\frac{1}{3}\right)^4 \cdot \frac{4^4}{4!}\right| = \frac{32}{243} \approx 0.132$.

6. (a) $f(3) = (x-2)^{-1}\big|_{x=3} = 1$
$f'(3) = -(x-2)^{-2}\big|_{x=3} = -1$
$f''(3) = 2(x-2)^{-3}\big|_{x=3} = 2$, so $\frac{f''(3)}{2!} = 1$
$f'''(3) = -6(x-2)^{-4}\big|_{x=3} = -6$, so $\frac{f'''(3)}{3!} = -1$
$f^{(n)}(3) = (-1)^n n!$, so $\frac{f^{(n)}(3)}{n!} = (-1)^n$
$f(x) = 1 - (x-3) + (x-3)^2 - (x-3)^3 + \cdots + (-1)^n(x-3)^n + \cdots$

(b) Integrate term by term.

$$\ln|x-2| = \int_3^x \frac{1}{t-2}\,dt$$

$$= \left[t - \frac{1}{2}(t-3)^2 + \frac{1}{3}(t-3)^3 - \frac{1}{4}(t-3)^4 + \cdots + (-1)^n \frac{(t-3)^{n+1}}{n+1} + \cdots \right]_3^x$$

$$= (x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} - \frac{(x-3)^4}{4} + \cdots + (-1)^n \frac{(x-3)^{n+1}}{n+1} + \cdots$$